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On 3-regular partitions with odd parts distinct

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Abstract We consider $\mathrm{pod}_3(n)$, the number of 3-regular partitions with odd parts distinct, whose generating function is

$$\sum_{n\geq 0} \mathrm{pod}_3(n) q^n = \frac{(-q;q^2)_\infty (q^6;q^6)_\infty}{(q^2;q^2)_\infty (-q^3;q^3)_\infty} = \frac{\psi(-q^3)}{\psi(-q)},$$

where

$$\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2} = \sum_{-\infty}^{\infty} q^{2n^2+n}.$$

For each $\alpha > 0$, we obtain the generating function for

$$\sum_{n\geq 0} \mathrm{pod}_3\left(3^\alpha n + \delta_\alpha\right) q^n,$$

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where $4\delta_\alpha \equiv -1 \pmod{3^\alpha}$ if α is even, $4\delta_\alpha \equiv -1 \pmod{3^{\alpha+1}}$ if α is odd.

We show that the sequence $\{\mathrm{pod}_3(n)\}$ satisfies the internal congruences

$$\mathrm{pod}_3(9n+2) \equiv \mathrm{pod}_3(n) \pmod{9}, \tag{0.1}$$

$$\mathrm{pod}_3(27n+20) \equiv \mathrm{pod}_3(3n+2) \pmod{27} \tag{0.2}$$

and

$$\mathrm{pod}_3(243n+182) \equiv \mathrm{pod}_3(27n+20) \pmod{81}. \tag{0.3}$$

Keywords 3-regular partitions · Odd parts distinct · Internal congruences

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1 Introduction

Let $\mathrm{pod}(n)$ denote the number of partitions of n in which odd parts are distinct (and even parts are unrestricted). The generating function of $\mathrm{pod}(n)$ is

$$\sum_{n\geq 0} \mathrm{pod}(n)q^n = \frac{1}{\psi(-q)}, \tag{1.1}$$

where

$$\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2}. \tag{1.2}$$

The function $\mathrm{pod}(n)$ appears in the works of Andrews [1,2], and Berkovich and Garvan [4]. Moreover, Berkovich and Garvan note that Andrews [3] considered a restricted version of $\mathrm{pod}(n)$ in which each part was required to be greater than 1. For the first time in 2011, Hirschhorn and Sellers [6] considered $\mathrm{pod}(n)$ from an arithmetic point of view and proved that for all $\alpha \geq 0$ and $n \geq 0$,

$$\mathrm{pod}\left(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod{3}. \tag{1.3}$$

By using modular forms, Radu and Sellers [7] established congruences modulo 5 and 7 for $\mathrm{pod}(n)$. For example,

$$\mathrm{pod}(135n+8,\ 107,\ 116) \equiv 0 \pmod{5}, \tag{1.4}$$

$$\mathrm{pod}(567n+260,\ 449) \equiv 0 \pmod{7}. \tag{1.5}$$

Recently, Cui et al. [5] derived congruences modulo 5 for $\mathrm{pod}(n)$ by using theta function identities. For example, for all $\alpha \geq 0$ and $n \geq 0$,

$$\mathrm{pod}\left(5^{2\alpha+4}n + \frac{11 \times 5^{2\alpha+3}+1}{8}\right) \equiv 0 \pmod{5}. \tag{1.6}$$

Our goal in this paper is to find arithmetic properties of the function $\mathrm{pod}_3(n)$, which enumerates the partitions of n into non-multiples of 3 in which the odd parts are distinct (and even parts unrestricted). The generating function of $\mathrm{pod}_3(n)$ is

$$\sum_{n\geq 0}\mathrm{pod}_3(n)q^n = \frac{\psi(-q^3)}{\psi(-q)}. \tag{1.7}$$

We will begin by showing that for $\alpha \geq 0$,

$$\sum_{n\geq 0}\mathrm{pod}_3\left(3^{2\alpha}n + \frac{3^{2\alpha}-1}{4}\right)q^n = \sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i}\,q^{i-1}\frac{\psi(-q^3)^{4i-3}}{\psi(-q)^{4i-3}} \tag{1.8}$$

and

$$\sum_{n\geq 0}\mathrm{pod}_3\left(3^{2\alpha+1}n + \frac{3^{2\alpha+2}-1}{4}\right)q^n = \sum_{i=1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,i}\,q^{i-1}\frac{\psi(-q^3)^{4i-1}}{\psi(-q)^{4i-1}}, \tag{1.9}$$

where the coefficient vectors $\boldsymbol{x}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, \cdots)$ are given by

$$\mathbf{x}_0 = (x_{0,1}, x_{0,2}, x_{0,3}, \cdots) = (1, 0, 0, \cdots), \tag{1.10}$$

and for $\alpha \geq 0$,

$$\mathbf{x}_{\alpha+1} = \boldsymbol{x}_\alpha A \quad \text{if } \alpha \text{ is even}, \tag{1.11}$$

$$\mathbf{x}_{\alpha+1} = \boldsymbol{x}_\alpha B \quad \text{if } \alpha \text{ is odd}, \tag{1.12}$$

where $A = (a_{i,j})_{i,j\geq 1}$ and $B = (b_{i,j})_{i,j\geq 1}$ are defined by

$$a_{i,j} = m_{4i-3,i+j-1}, \quad b_{i,j} = m_{4i-1,i+j-1}, \tag{1.13}$$

where $M = (m_{i,j})_{i,j\geq 1}$ is defined as follows: The first three rows of M are

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{1.14}$$

and

$$\text{for } i \geq 4,\ m_{i,1} = 0, \text{ and for } j \geq 2,\ m_{i,j} = 3m_{i-1,j-1} + 3m_{i-2,j-1} + m_{i-3,j-1}. \tag{1.15}$$

We will calculate $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ and enough of $\mathbf{x}_5$ to show that

$$\mathrm{pod}_3(9n+2) \equiv \mathrm{pod}_3(n) \pmod 9, \tag{1.16}$$

$$\mathrm{pod}_3(27n+20) \equiv \mathrm{pod}_3(3n+2) \pmod{27} \tag{1.17}$$

and

$$\mathrm{pod}_3(243n+182) \equiv \mathrm{pod}_3(9n+2) \pmod{81}. \tag{1.18}$$

2 Preliminaries

It is easy to verify that

$$\psi(q) = P(q^3) + q\psi(q^9), \tag{2.1}$$

where

$$P(q) = \sum_{-\infty}^{\infty} q^{(3n^2+n)/2}. \tag{2.2}$$

Also, with ω a cube root of unity other than 1,

$$\psi(q)\psi(\omega q)\psi(\omega^2 q) = \frac{\psi(q^3)^4}{\psi(q^9)} = P(q^3)^3 + q^3\psi(q^9)^3, \tag{2.3}$$

so

$$P(q^3)^3 = \frac{\psi(q^3)^4 - q^3\psi(q^9)^4}{\psi(q^9)}. \tag{2.4}$$

If we set $-q$ for q in (2.1) and (2.4), we find

$$\psi(-q) = P(-q^3) - q\psi(-q^9) \tag{2.5}$$

and

$$P(-q^3)^3 = \frac{\psi(-q^3)^4 + q^3\psi(-q^9)^4}{\psi(-q^9)}. \tag{2.6}$$

Now let

$$\zeta = \frac{\psi(-q)}{q\psi(-q^9)}, \quad \rho = \frac{P(-q^3)}{q\psi(-q^9)}, \quad T = \frac{\psi(-q^3)^4}{q^3\psi(-q^9)^4}. \tag{2.7}$$

Then, from (2.5) and (2.7),

$$\zeta = \frac{\psi(-q)}{q\psi(-q^9)} = \frac{P(-q^3) - q\psi(-q^9)}{q\psi(-q^9)} = -1 + \frac{P(-q^3)}{q\psi(-q^9)} = -1 + \rho, \tag{2.8}$$

$$\zeta^2 = 1 - 2\rho + \rho^2, \tag{2.9}$$

and from (2.6) to (2.8)

$$
\begin{aligned}
\zeta^3 &= -1 + 3\rho - 3\rho^2 + \rho^3 \\
&= -1 + 3\rho - 3\rho^2 + \frac{P(-q^3)^3}{q^3\psi(-q^9)^3} \\
&= -1 + 3\rho - 3\rho^2 + \frac{\psi(-q^3)^4 + q^3\psi(-q^9)^4}{q^3\psi(-q^9)^4} \\
&= -1 + 3\rho - 3\rho^2 + (T+1) \\
&= 3\rho - 3\rho^2 + T.
\end{aligned} \tag{2.10}
$$

It follows from (2.8) to (2.10) that

$$\zeta^3 + 3\zeta^2 + 3\zeta = T. \tag{2.11}$$

We can write (2.11)

$$\frac{1}{\zeta} = \frac{1}{T}(3 + 3\zeta + \zeta^2), \tag{2.12}$$

so

$$\frac{1}{\zeta^i} = \frac{1}{T}\left(\frac{3}{\zeta^{i-1}} + \frac{3}{\zeta^{i-2}} + \frac{1}{\zeta^{i-3}}\right). \tag{2.13}$$

Now let H be the "huffing" operator modulo 3, that is,

$$H\left(\sum a_n q^n\right) = \sum a_{3n} q^{3n}. \tag{2.14}$$

If we apply H to (2.13), we find

$$H\left(\frac{1}{\zeta^i}\right) = \frac{1}{T}\left(3H\left(\frac{1}{\zeta^{i-1}}\right) + 3H\left(\frac{1}{\zeta^{i-2}}\right) + H\left(\frac{1}{\zeta^{i-3}}\right)\right). \tag{2.15}$$

Now,

$$H(\zeta^2) = H(1 - 2\rho + \rho^2) = 1, \tag{2.16}$$

$$H(\zeta) = H(-1 + \rho) = -1, \tag{2.17}$$

$$H(1) = 1, \tag{2.18}$$

$$H\left(\frac{1}{\zeta}\right) = \frac{1}{T}(3 - 3 + 1) = \frac{1}{T}, \tag{2.19}$$

$$H\left(\frac{1}{\zeta^2}\right) = \frac{1}{T}\left(\frac{3}{T} + 3 - 1\right) = \frac{2}{T} + \frac{3}{T^2}, \tag{2.20}$$

$$H\left(\frac{1}{\zeta^3}\right) = \frac{1}{T}\left(3\left(\frac{2}{T} + \frac{3}{T^2}\right) + 3\left(\frac{1}{T}\right) + 1\right) = \frac{1}{T} + \frac{9}{T^2} + \frac{9}{T^3}. \tag{2.21}$$

and so on.

Indeed, for $i \geq 1$, we can write

$$H\left(\frac{1}{\zeta^i}\right) = \sum_{j=1}^{i} \frac{m_{i,j}}{T^j}, \tag{2.22}$$

where the $m_{i,j}$ are defined in (1.8) and (1.9).

The $m_{i,j}$ form a matrix M, the first nine rows of which are

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 10 & 36 & 27 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 60 & 135 & 81 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 54 & 297 & 486 & 243 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 28 & 378 & 1323 & 1701 & 729 & 0 & 0 & \cdots \\ 0 & 0 & 8 & 306 & 2160 & 5508 & 5832 & 2187 & 0 & \cdots \\ 0 & 0 & 1 & 162 & 2349 & 10935 & 21870 & 19683 & 6561 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.23}$$

It is clear that if $i > 3j$ then $m_{i,j} = 0$. So, $m_{4i-3,j} = 0$ if $j < i$.

It follows that we can write

$$H\left(\frac{1}{\zeta^{4i-3}}\right) = \sum_{j=i}^{4i-3} \frac{m_{4i-3,j}}{T^j} = \sum_{j=1}^{3i-2} \frac{m_{4i-3,i+j-1}}{T^{i+j-1}} = \sum_{j=1}^{3i-2} \frac{a_{i,j}}{T^{i+j-1}}, \tag{2.24}$$

where the $a_{i,j}$ are defined in (1.7).

The $a_{i,j}$ form a matrix A, the first three rows of which are

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 60 & 135 & 81 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 162 & 2349 & 10935 & 21870 & 19683 & 6561 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.25}$$

Similarly, $m_{4i-1,j} = 0$ if $j < i$, so we can write

$$H\left(\frac{1}{\zeta^{4i-1}}\right) = \sum_{j=i}^{4i-1} \frac{m_{4i-1,j}}{T^j} = \sum_{j=1}^{3i} \frac{m_{4i-1,i+j-1}}{T^{i+j-1}} = \sum_{j=1}^{3i} \frac{b_{i,j}}{T^{i+j-1}}, \tag{2.26}$$

where the $b_{i,j}$ are defined in (1.7).

The $b_{i,j}$ form a matrix B, the first two rows of which are

$$B = \begin{pmatrix} 1 & 9 & 9 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 28 & 378 & 1323 & 1701 & 729 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.27}$$

We can write (2.24)

$$H\left(\left(\frac{q\psi(-q^9)}{\psi(-q)}\right)^{4i-3}\right) = \sum_{j=1}^{3i-2} a_{i,j} \left(\frac{q^3\psi(-q^9)^4}{\psi(-q^3)^4}\right)^{i+j-1}, \tag{2.28}$$

and this can be rearranged to

$$H\left(q^{i-3}\frac{\psi(-q^3)^{4i-3}}{\psi(-q)^{4i-3}}\right) = \sum_{j=1}^{3i-2} a_{i,j} q^{3j-3}\frac{\psi(-q^9)^{4j-1}}{\psi(-q^3)^{4j-1}}. \tag{2.29}$$

Similarly, (2.26) is

$$H\left(\left(\frac{q\psi(-q^9)}{\psi(-q)}\right)^{4i-1}\right) = \sum_{j=1}^{3i} b_{i,j} \left(\frac{q^3\psi(-q^9)^4}{\psi(-q^3)^4}\right)^{i+j-1}, \tag{2.30}$$

and this can be rearranged to

$$H\left(q^{i-1}\frac{\psi(-q^3)^{4i-1}}{\psi(-q)^{4i-1}}\right) = \sum_{j=1}^{3i} b_{i,j} q^{3j-3}\frac{\psi(-q^9)^{4j-3}}{\psi(-q^3)^{4j-3}}. \tag{2.31}$$

3 Proof

Here we prove (1.8) and (1.9).

We see that (1.8) holds for $\alpha = 0$.

Suppose (1.8) holds for some $\alpha \geq 0$. Then

$$\sum_{n\geq 0} \mathrm{pod}_3\left(3^{2\alpha}n + \frac{3^{2\alpha}-1}{4}\right) q^{n-2} = \sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i}\, q^{i-3}\frac{\psi(-q^3)^{4i-3}}{\psi(-q)^{4i-3}}. \tag{3.1}$$

If we apply the operator H to (3.1), we find

$$\sum_{n\geq 0}\mathrm{pod}_3\left(3^{2\alpha}(3n+2)+\frac{3^{2\alpha}-1}{4}\right)q^{3n} \tag{3.2}$$
$$=\sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i}\,H\left(q^{i-3}\frac{\psi(-q^3)^{4i-3}}{\psi(-q)^{4i-3}}\right)$$
$$=\sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i}\sum_{j=1}^{3i-2} a_{i,j}q^{3j-3}\frac{\psi(-q^9)^{4j-1}}{\psi(-q^3)^{4j-1}}$$
$$=\sum_{j=1}^{(3^{2\alpha+1}+1)/4}\left(\sum_{i=1}^{(3^{2\alpha}+3)/4} x_{2\alpha,i}a_{i,j}\right)q^{3j-3}\frac{\psi(-q^9)^{4j-1}}{\psi(-q^3)^{4j-1}}$$
$$=\sum_{j=1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,j}q^{3j-3}\frac{\psi(-q^9)^{4j-1}}{\psi(-q^3)^{4j-1}}.$$

If we now replace q^3 by q, we obtain

$$\sum_{n\geq 0}\mathrm{pod}_3\left(3^{2\alpha+1}n+\frac{3^{2\alpha+2}-1}{4}\right)q^{n}=\sum_{j=1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,j}q^{j-1}\frac{\psi(-q^3)^{4j-1}}{\psi(-q)^{4j-1}}, \tag{3.3}$$

which is (1.9).

Now suppose (1.9) holds for some $\alpha\geq 0$. That is,

$$\sum_{n\geq 0}\mathrm{pod}_3\left(3^{2\alpha+1}n+\frac{3^{2\alpha+2}-1}{4}\right)q^{n}=\sum_{i=1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,i}q^{i-1}\frac{\psi(-q^3)^{4i-1}}{\psi(-q)^{4i-1}}. \tag{3.4}$$

If we apply the operator H, we find

$$\sum_{n\geq 0}\mathrm{pod}_3\left(3^{2\alpha+1}(3n)+\frac{3^{2\alpha+2}-1}{4}\right)q^{3n} \tag{3.5}$$
$$=\sum_{i\geq 1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,i}\,H\left(q^{i-1}\frac{\psi(-q^3)^{4i-1}}{\psi(-q)^{4i-1}}\right) \tag{3.6}$$
$$=\sum_{i=1}^{(3^{2\alpha+1}+1)/4} x_{2\alpha+1,i}\sum_{j=1}^{3i} b_{i,j}q^{3j-3}\frac{\psi(-q^9)^{4j-3}}{\psi(-q^3)^{4j-3}} \tag{3.7}$$
$$=\sum_{j=1}^{(3^{2\alpha+2}+3)/4}\left(\sum_{i=1}^{(3^{2\alpha+1}+3)/4} x_{2\alpha+1,i}b_{i,j}\right)q^{3j-3}\frac{\psi(-q^9)^{4j-3}}{\psi(-q^3)^{4j-3}} \tag{3.8}$$

$$= \sum_{j=1}^{(3^{2\alpha+2}+3)/4} x_{2\alpha+2,j} q^{3j-3} \frac{\psi(-q^9)^{4j-3}}{\psi(-q^3)^{4j-3}}. \tag{3.9}$$

If we now replace q^3 by q, we obtain

$$\sum_{n\geq 0} \mathrm{pod}_3 \left(3^{2\alpha+2} n + \frac{3^{2\alpha+2}-1}{4}\right) q^n = \sum_{j=1}^{(3^{2\alpha+2}+3)/4} x_{2\alpha+2,j} q^{j-1} \frac{\psi(-q^3)^{4j-3}}{\psi(-q)^{4j-3}},$$

which is (1.8) with $\alpha+1$ for α. This completes the proof of (1.8) and (1.9) by induction.

4 Congruences

Let $\nu(N)$ be the (highest) power of 3 that divides N.

It is not hard to show that

$$\nu(m_{i,j}) \geq \frac{3j-i-2}{2}, \tag{4.1}$$

$$\nu(a_{i,j}) \geq \frac{3j-i-2}{2} \tag{4.2}$$

and

$$\nu(b_{i,j}) \geq \frac{3j-i-4}{2}, \tag{4.3}$$

and hence that

$$\nu(x_{2\alpha,i}) \geq \frac{3i-5}{2}, \quad \nu(x_{2\alpha+1}, i) \geq \frac{3i-3}{2}. \tag{4.4}$$

The first six $\mathbf{x}_\alpha$ are

$$\mathbf{x}_0 = (1, 0, 0, 0, 0, 0, 0, 0, \cdots), \tag{4.5}$$

$$\mathbf{x}_1 = (1, 0, 0, 0, 0, 0, 0, 0, \cdots), \tag{4.6}$$

$$\mathbf{x}_2 = (1, 9, 9, 0, 0, 0, 0, 0, \cdots), \tag{4.7}$$

$$\mathbf{x}_3 = (55, 74 \times 3^3, 92 \times 3^5, 136 \times 3^6, 10 \times 3^9, 3^{11}, 3^{10}, 0, 0, \cdots) \tag{4.8}$$

$$\begin{aligned}\mathbf{x}_4 = (&55, 44611 \times 3^2, 14511529 \times 3^2, 17777951 \times 3^6, 94056 \times 3^8, 867259217 \times 3^9, \\ &570794317 \times 3^{12}, 768525037 \times 3^{14}, 2199360935 \times 3^{15}, 508504163 \times 3^{18}, \\ &261164933 \times 3^{20}, 905255326 \times 3^{20}, 29281 \times 3^{24}, 6464449 \times 3^{26}, \\ &3240631 \times 3^{27}, 135470 \times 3^{30}, 12530 \times 3^{32}, 2485 \times 3^{33}, \\ &37 \times 3^{36}, 3^{38}, 3^{36}, 0, 0, \cdots)\end{aligned} \tag{4.9}$$

and

$$\mathbf{x}_5 = (132611311, 1065245364491 \times 3^3, 1110525359080559 \times 3^5, \\ 761677579443625648 \times 3^6, 224549464126506021110 \times 3^9, \\ 101352476270091196401 1 \times 3^{11}, 739170712762862370442615 \times 3^{10}, \cdots). \tag{4.10}$$

It follows from (4.5) and (4.7) that

$$\mathrm{pod}_3(9n+2) \equiv \mathrm{pod}_3(n) \pmod 9, \tag{4.11}$$

from (4.6) and (4.8) that

$$\mathrm{pod}_3(27n+20) \equiv \mathrm{pod}_3(3n+2) \pmod{27} \tag{4.12}$$

and from (4.8) and (4.10) that

$$\mathrm{pod}_3(243n+182) \equiv \mathrm{pod}_3(27n+20) \pmod{81}. \tag{4.13}$$

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